SMOOTHED PARTICLE HYDRODYNAMICS

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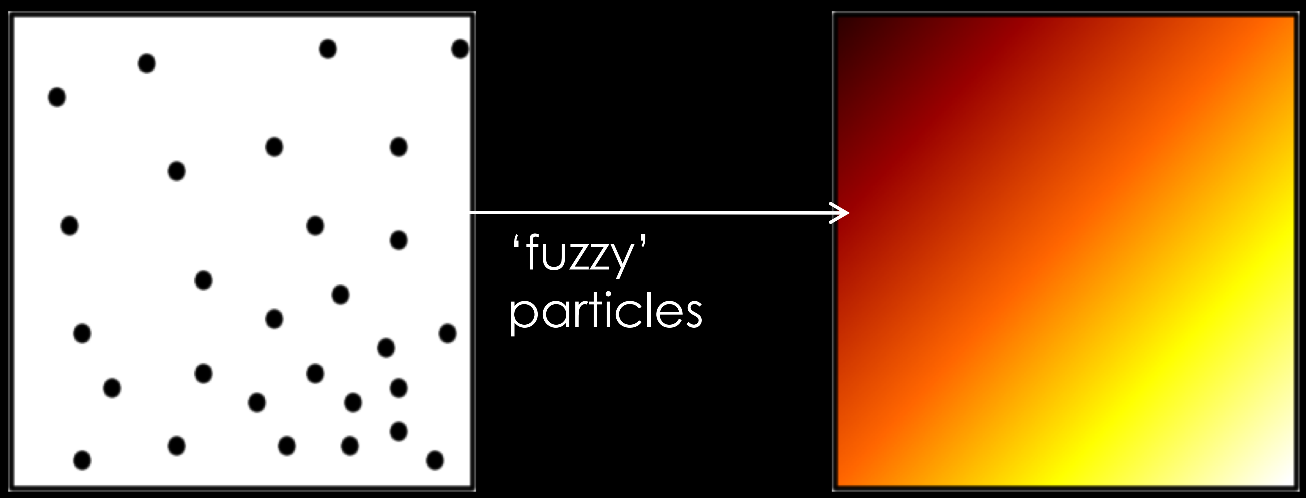
SO, SMOOTHED PARTICLES…

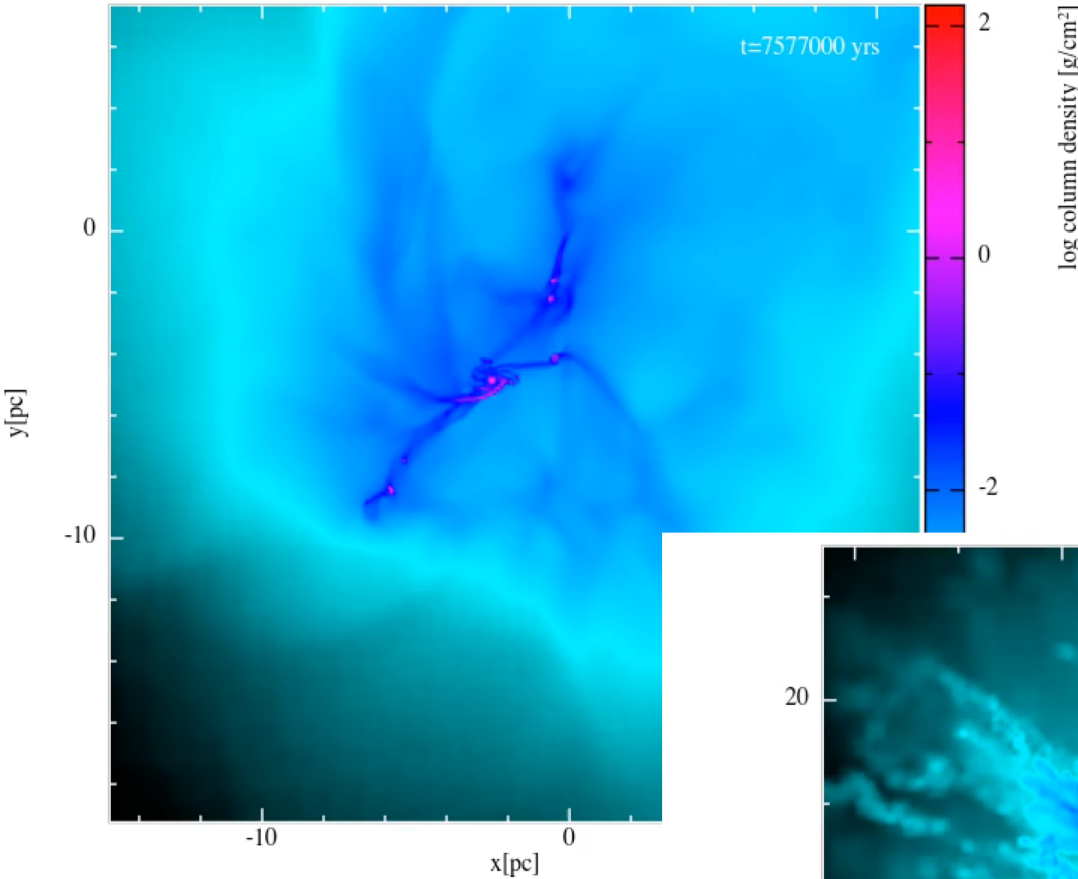
- Developed initially by Lucy (1977) and Gingold & Monaghan (1977).
- SPH represents a continuous fluid as a system of discrete particles, each with a mass and ‘extent’.
- This means the method is Lagrangian – our measurement points, the grid, are the particles, and they move with the flow.
- Originally developed in astrophysics, but now used in many fields.

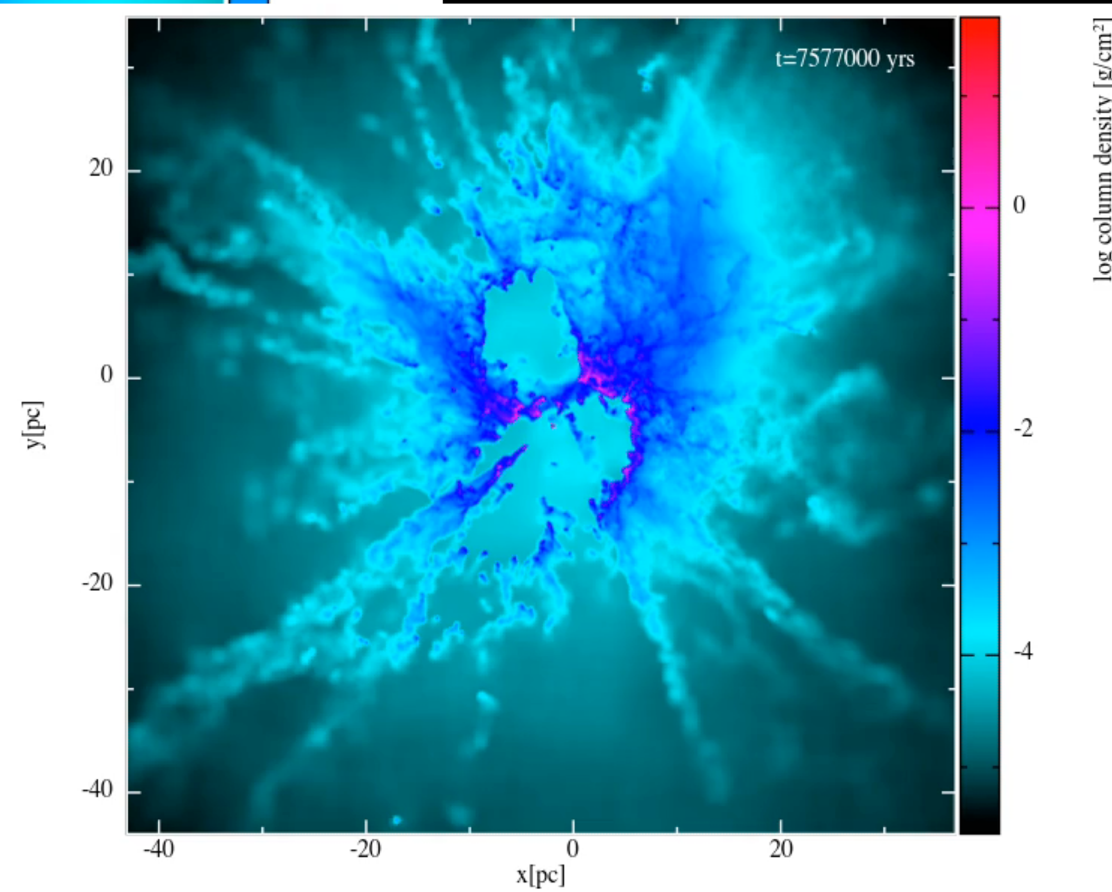
SO, SMOOTHED PARTICLES…

- In two dimensions, with uniform mass particles increasing in density towards the lower right:



(very crudely)





Lucas et al., in prep.

THE BASIS OF SPH I

- SPH uses interpolation: any quantity $f(\mathbf{r})$ can be smoothed using a kernel $W(\mathbf{r},h)$ where h is the smoothing length.
- At any point the smoothed value of f is then

$$\langle f(\mathbf{r})\rangle = \int_V f(\mathbf{r}')W(\mathbf{r}-\mathbf{r}',h)\,\mathrm{d}^3\mathbf{r}'$$

- The kernel must normalise and as h becomes vanishingly small:

$$\lim_{h\to 0} W(\mathbf{r}-\mathbf{r}',h) = \delta(\mathbf{r}-\mathbf{r}')$$

$$\lim_{h\to 0}\langle f(\mathbf{r})\rangle = f(\mathbf{r})$$

THE BASIS OF SPH II

- But it's still an integral…
- Convert to a sum over particles by replacing the infinitesimal volume element with the volume for a particle *b*:

$$\Delta V_b = \frac{m_b}{\rho_b}$$

- Then:

$$\langle f(\mathbf{r}) \rangle = \sum_{b=1}^{N} \frac{m_b}{\rho_b} f(\mathbf{r}_b) W(|\mathbf{r} - \mathbf{r}_b|, h)$$

THE BASIS OF SPH III

- We also want the gradient of *f*. From integration by parts:

$$\langle \nabla f(\mathbf{r}) \rangle = \int_S f(\mathbf{r}') W(|\mathbf{r}-\mathbf{r}'|, h) \hat{\mathbf{n}} \, \mathrm{d}S - \int_V f(\mathbf{r}') \nabla W(|\mathbf{r}-\mathbf{r}'|, h) \, \mathrm{d}^3\mathbf{r}'$$

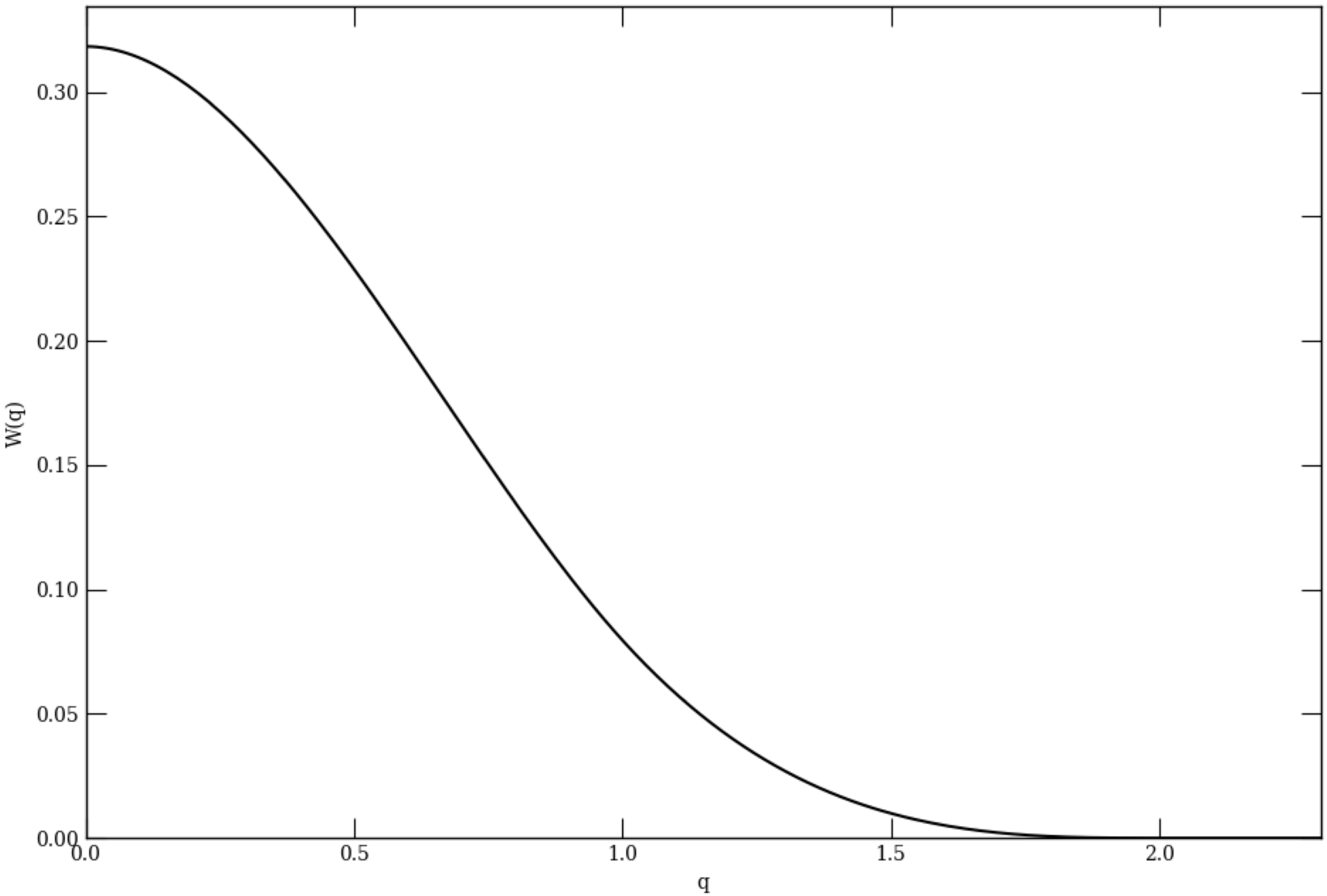
- If the kernel is compact – it goes to 0 beyond some distance – then the surface term also becomes 0.
- Changing to use the gradient with respect to **r** rather than **r**’ and switching to summation as before gives:

$$\langle \nabla f(\mathbf{r}) \rangle = \sum_{b=1}^{N} \frac{m_b}{\rho_b} f(\mathbf{r}_b) \nabla_r W(|\mathbf{r}-\mathbf{r}_b|, h)$$

THE BASIS OF SPH IV

- The kernel should be peaked and even (Benz 1990) like a Gaussian to ensure small $O(h^2)$ errors.
- The kernel should be compact (previous slide).
- Very often a cubic spline (Monaghan & Lattanzio 1985) is used. For a 3D kernel, with $q = r / h$, this is:

$$W(r,h) = \frac{1}{\pi h^3} \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & \text{where } 0 \le q < 1, \\ \frac{1}{4}(2-q)^3 & \text{where } 1 \le q < 2, \\ 0 & \text{elsewhere} \end{cases}$$

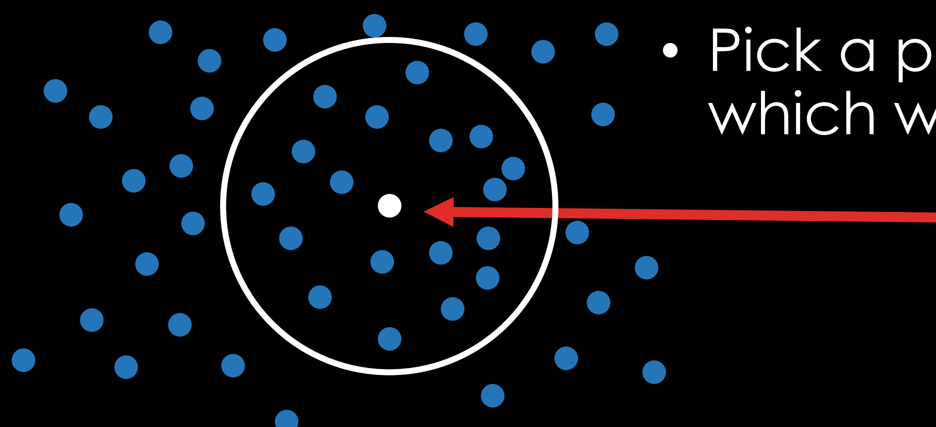

0.30
0.25
0.20
0.15
0.10
0.05
0.00
W(q)
0.0
0.5
1.0
1.5
2.0
q

THE BASIS OF SPH V

- The choice of kernel is important – having a peaked, even, Gaussian like function means that the error is small (second order in *h*) – we can drop the angled brackets from now on.
- And making compact means that we don't have to do the sum over any particles outside the kernel as they contribute nothing!
- Find nearby 'neighbour' particles, and that's all you need.
- Choose *h*'s value to give ~50 neighbours (not too many, not too few). For astrophysics, we also let *h* vary throughout the fluid.

SPH CONDENSED

- SPH particles represent a fluid element with mass *m* and a kernel to make it fuzzy with smoothing length *h*.



- Pick a particle which we'll call *a*

- The kernel extends as far as 2*h* and contains N_{neigh} neighbour particles within that distance.
- The symmetrised kernel (mean *h* or mean *W*) between *a* and any neighbour *b* is called W_{ab}.

SPH CONDENSED

- Calculate any smoothed quantity for *a* by looping over neighbours *b* smoothed by the kernel with

$$f_a = \sum_b \frac{m_b}{\rho_b} f_b W_{ab}$$

- We're just summing a lot of overlapping not-quite-Gaussians.
- Calculate the smoothed gradient of the same with

$$\nabla f_a = \sum_b \frac{m_b}{\rho_b} f_b \nabla_a W_{ab}$$

FLUID EQUATIONS I
MASS CONTINUITY

- Substitute the density into the SPH equation:

$$\rho_a = \sum_b m_b W_{ab}$$

- Or as a time derivative, with $\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$:

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab}$$

- Automatically conserves mass!

FLUID EQUATIONS II
MOMENTUM EQUATION

- The Lagrangian momentum equation with no external forces is

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}$$

- From here we can get for particle *a*, using an identity for the pressure gradient and bringing in the SPH gradient equation (see Monaghan 1992)

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} \right) \nabla_a W_{ab}$$

- Momentum conservation is guaranteed!

FLUID EQUATIONS III
ENERGY EQUATION

- Slightly differently, look at the Eulerian energy equation:

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla) u = -\frac{P}{\rho} \nabla \cdot \mathbf{v}$$

- Multiply by the kernel and integrate over its volume (as in Benz 1990) to get

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\rho_a^2} \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab}$$

- Summing the total change in energy over all particles again shows that it is conserved.

FLUID EQUATIONS III
ENERGY EQUATION

- Alternatively, start with the Lagrangian energy equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla \cdot \mathbf{v}$$

- This can be combined with the previous form to obtain

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{1}{2}\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right) \mathbf{v}_{ab} \cdot \nabla_a W_{ab}$$

- This has explicit conservation of energy between particle pairs, but it can be possible to get negative energies so our code (Benz 1990) uses the previous form.

THE SMOOTHING LENGTH

- If *h* is allowed to vary, then how is it calculated?
- For 'standard' SPH it's evolved alongside the other quantities.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{3} h \nabla \cdot \mathbf{v}$$

- Alternatively, we can use more modern 'grad-h SPH' which properly takes into account *h*'s spatial variation. This can be more complex and also requires changes to the momentum and energy equations.

FINAL POINTS

- The above equations are for an inviscid fluid – shocks can't form!
- This means an extra artificial viscosity Π_{ab} has to be calculated and included in the momentum and energy equations to allow for dissipation.
- To run a simulation, you really only need three more things:
  - An equation of state to relate pressure, temperature, density, internal energy…
  - An integrator to take the time derivatives and move particles forwards in time (leapfrog, Runge-Kutta…)
  - Your initial conditions!

EXAMPLES

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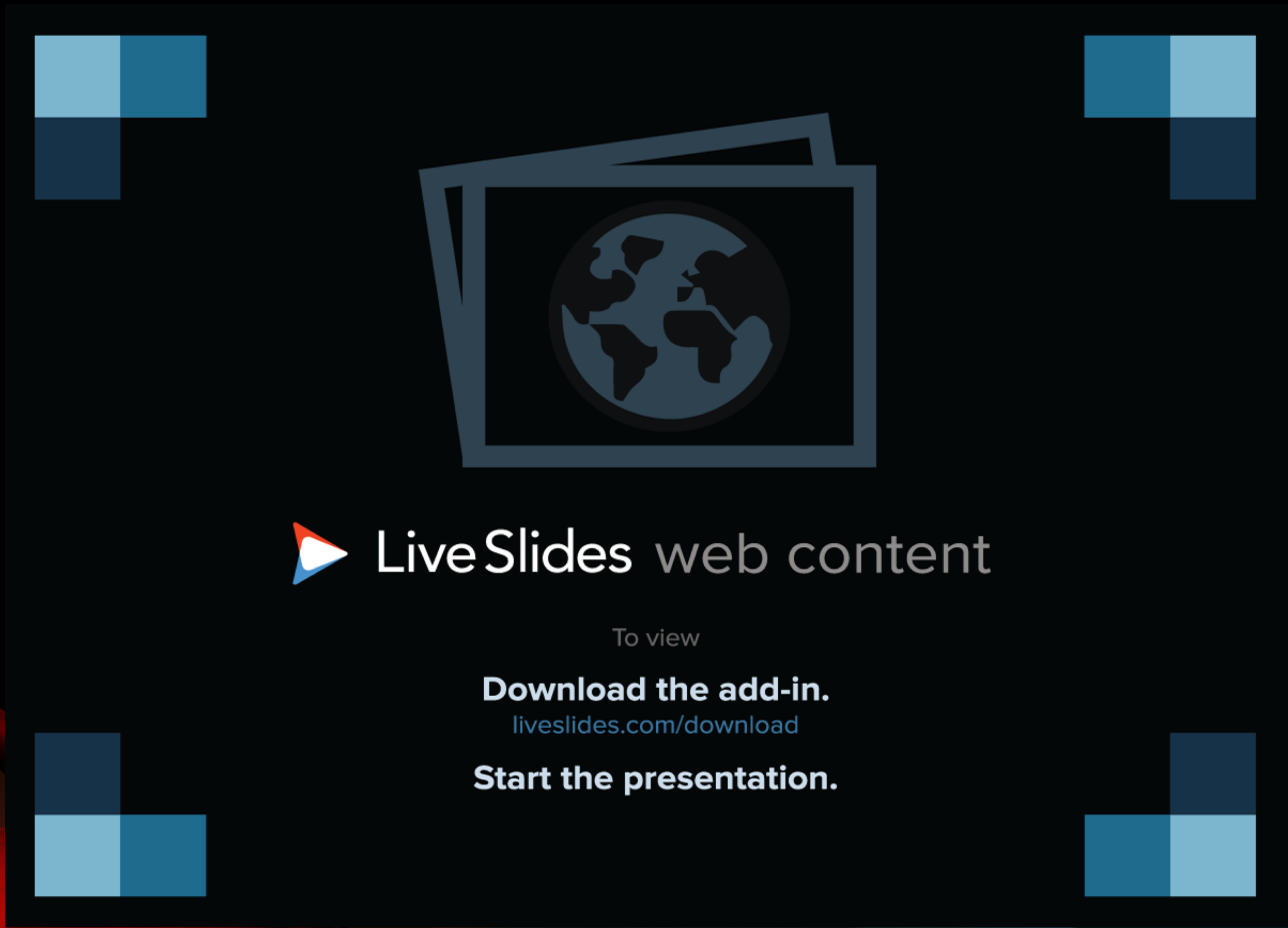
Bate M.R., 2018, astro-ph/1801.07721

Mooring test with DualSPHysics



USEFUL RESOURCES

- Benz W., 1990, in Proceedings of the NATO Advanced Research Workshop on the Numerical Modelling of Nonlinear Stellar Pulsations, ed. J.R. Buchler, Vol 302, 269 (it's in the library - QB812.N8)
- Monaghan J.J., 1992, Annual Review of Astronomy & Astrophysics, 30, 543 (http://adsabs.harvard.edu/abs/1992ARA%26A..30..543M)
- Monaghan J.J., 2005, Reports on Progress in Physics, 68, 1703 (http://adsabs.harvard.edu/abs/2005RPPh...68.1703M)
- Rosswog S., 2009, New Astronomy Reviews, 53, 78 (http://adsabs.harvard.edu/abs/2009NewAR..53...78R)
- Springel V., 2010, Annual Review of Astronomy & Astrophysics, 48, 391 (http://adsabs.harvard.edu/abs/2010ARA%26A..48..391S)
- Price D.J., 2012, Journal of Computational Physics, 231, 759
- Chapter 3 of Daniel Price's thesis (https://arxiv.org/abs/astro-ph/0507472)

CODES YOU CAN USE

- Phantom: https://bitbucket.org/danielprice/phantom/wiki/Home
- SWIFT: http://icc.dur.ac.uk/swift/
- GANDALF: http://gandalfcode.github.io/
- GADGET: https://wwwmpa.mpa-garching.mpg.de/gadget/
- DualSPHysics: http://dual.sphysics.org/
- PySPH: https://github.com/pypr/pysph

ARTIFICIAL VISCOSITY I

- One of SPH's major problems: by default, it doesn't do shocks! There's no dissipation, entropy is constant.
- We need to bring dissipation in somehow, so we introduce artificial viscosity (Monaghan & Gingold 1983):

$$\Pi_{ab} = \begin{cases} \frac{-\alpha \bar{c}_{ab}\mu_{ab} + \beta\mu_{ab}^2}{\bar{\rho}_{ab}} & \text{where } \mathbf{v}_{ab}\cdot\mathbf{r}_{ab} < 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_{ab} = \frac{h\mathbf{v}_{ab}\cdot\mathbf{r}_{ab}}{\mathbf{r}_{ab}^2 + \eta^2}$$

ARTIFICIAL VISCOSITY II

- This allows for the conversion of kinetic to thermal energy.
- The momentum and energy equations become

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab}$$

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{P_a}{\rho_a^2} \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab} + \frac{1}{2} \sum_b m_b \Pi_{ab} \mathbf{v}_{ab} \cdot \nabla_a W_{ab}$$

ARTIFICIAL VISCOSITY III

- Very often $\alpha = 1$ and $\beta = 2$ – these control the strength of the viscosity.
- The main problem is making sure that the artificial viscosity doesn't do much (or anything!) away from shocks - a problem when you have shear flows.
- Options are to use a switch (Balsara 1995) or to make α a variable which grows in shocks and then with time decays away (Morris & Monaghan 1997).
- SPH is still smoothed – discontinuities don't exist and a shock will be a transition over about $3h$.